Math 246C Lecture 6 Notes

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1 The Monodromy Theorem and Application to Linear ODE

1.1 The monodromy theorem

Last time, we introduced the notion of analytic continuation. If $a \in X$, and $f_a \in O_a$, then an analytic continuation along some curve $\gamma : [0,1] \to X$ is a lift $\tilde{\gamma}$ to the sheaf of germs such that for all $t \in [0,1]$, $\tilde{\gamma}(t) = f_{\gamma(t)} \in O_{\gamma(t)}$, and $t \mapsto f_{\gamma(t)}$ is continuous.

$$\begin{tikzcd} & O_X \arrow[d, "p"] \\ Z \arrow[ur, dashed, "\tilde{\gamma}"] \arrow[r, "\gamma"'] & X \end{tikzcd}$$

That is, for all $t_0 \in [0,1]$, there is a neighborhood $I_{t_0} \subseteq [0,1]$ of t_0 and an open set $\omega \subseteq X$ such that $\gamma(I_{t_0}) \subseteq \omega$, and $\tilde{f} \in \mathrm{Hol}(\omega)$: $\tilde{f}_{\gamma(t)} = f_{\gamma(t)}$ for all $t \in I_{t_0}$.

Theorem 1.1 (monodromy theorem)**.** *Let X be a Riemann surface, let $a, b \in X$, and let γ_0, γ_1 be homotopic curves from a to b. Let $f_a \in O_a$. Let $H(t,s)$ be a homotopy between γ_0 and γ_1, and assume that f_a has an analytic continuation $\tilde{\gamma}_s$ along $\gamma_s(t) = H(t,s)$ for all s. Then $s \mapsto \gamma_s(1) \in O_b$ are equal for all s. In particular, $\tilde{\gamma}_0(1) = \tilde{\gamma}_1(1)$.*

Proof. Apply the homotopy lifting theorem to the local homeomorphism $p : O_X \to X$. $\square$

Corollary 1.1. *Let X be a simply connected Riemann surface, and let $a \in X$. Let $f_a \in O_a$. be a holomorphic germ which can be continued along any curve starting at a. Then there exists a unique globally defined holomorphic function $F \in \mathrm{Hol}(X)$ such that $F_a = f_a$ in O_a.*

Proof. When $x \in X$, let γ be a path from a to x, and let $f_x \in O_x$ be the analytic continuation of f_a along γ (f_x is independent of the choice of γ). Define $F(x) = f_x(x)$. $\square$

1.2 Linear ODE in the complex domain

Here is the historical origin of the idea of monodromy. This will be a good example of the applications of our theory.

Proposition 1.1. *Let $\Omega \subseteq \mathbb{C}$ be open, and let $A \in \mathrm{Hol}(\Omega, \mathrm{Mat}_{n\times n}(\mathbb{C}))$. Let Ω be simply connected. Then for all $z_0 \in \Omega$ and $x_0 \in \mathbb{C}^n$, then Cauchy problem*

$$x'(z) = A(z)x(z), \qquad x(z_0) = x_0$$

has a unique solution $x(z) \in \mathrm{Hol}(\Omega, \mathbb{C}^n)$

Proof. (idea) Write

$$x(z) = x_0 + \int_{\gamma_{z_0,z}} A(\zeta)x(\zeta)\,d\zeta,$$

and solve the integral equation by Picard's iterations. □

Assume now that $\Omega = \{0 < |z| < 1\}$ is not simply connected. We have the covering map $e^\zeta : \{\mathrm{Re}(\zeta) < 0\} \to \{0 < |z| < 1\}$, and we can lift the ODE to $\{\mathrm{Re}(\zeta) < 0\}$. If we let $y(\zeta) = z(e^\zeta)$, then

$$y'(\zeta) = \underbrace{e^\zeta A(e^\zeta)}_{2\pi i\text{-periodic}} y(\zeta).$$

We argue more directly: Let $\omega \subseteq \Omega$ be a small, simply connected neighborhood of $z_0 \in \{0 < |z| < 1\}$, and let $V(\omega) = \{x(z) \in \mathrm{Hol}(\omega, \mathbb{C}^n) : x'(z) = A(z)z(z) \text{ in } \omega\}$. This is an n-dimensional vector space. We can continue elements of $V(\omega)$ analytically: let $\Gamma_1 = \{z \in \Omega\text{''}\alpha < \arg(z) < \beta\}$ with $\alpha < 0$, $\beta > \pi$, and $\Gamma_1 \supseteq \omega$. Then $V(\Gamma_1)$ is the set of solutions to the ODE in Γ_1. We have the extension map $E : V(\omega) \to V(\Gamma_1)$. We then restrict to a domain ω' on the other side of the disc, extend to another sector Γ_2, and restrict to ω. We get a linear bijective map $S : V(\omega) \to V(\omega)$ called the **monodromy map** of this ODE.

Let $x_1, \dots, x_n$ be a basis for $V(\omega)$, and let $F(z) = \begin{bmatrix} x_1(z) & \cdots & x_n(z) \end{bmatrix}$ be the fundamental matrix with columns x_i. Write

$$Sx_j(z) = \sum_k S_{k,j} x_k(z).$$

If we denote $x_1(ze^{2\pi i}) = Sx_j(z)$, we get

$$F(ze^{2\pi i}) = F(z)A$$

for $z \in \omega$. We claim that there exists a matrix C such that $F(z) = Q(z)z^C$ in ω, where $Q(z) \in \mathrm{Hol}(0 < |z| < 1)$ and $z^C = e^{C\log(z)}$. To get the claim, we write $S = e^2\pi iC$ and check that $Q(z)$ satisfies $Q(ze^{2\pi i}) = Q(z)$.

1.3 Analytic continuation to larger Riemann surfaces

Let X be a Riemann surface, and let $\varphi \in O_a$ for some $a \in X$. We would like to construct a new Riemann surface which arises by analytic continuation of φ.

Definition 1.1. An **analytic continuation** of φ is given by (Y, p, f, b), where Y is a Riemann surface, $p : Y \to X$ is holomorphic with no ramification points, $f \in \mathrm{Hol}(Y)$, $b \in p^{-1}(a)$, and $f_b = p^*(\varphi)$. Here, p^* is the pullback map $p^*(\varphi) = \varphi \circ p$.